MODULE 4: RATIONAL NUMBERS, PART I:

COMMON FRACTIONS

In the last module we mentioned that rational numbers arose as the quotient of one whole number divided by any non-zero whole number. There are many ways to motivate rational numbers, but none is simpler than the way in which youngsters share, for example, candy equally. Most likely you are aware of the counting technique: "1 for me, 1 for you. 1 for me, 1 for you...." If there were, say, 8 pieces of candy, the youngsters could divide the amount by 2 *without knowing how to divide.*

3 ÷ 0 means the number we must multiply by 0 to get 3. Since 0 times any number is 0, no such number exists.

In fact, as we shall see below, they wouldn't even hav to know that there were 8 pieces.

To see this, let's use a tally mark to represent a piece of candy and we'll put a slash mark (/) through any tally mark representing a piece we give away. So the procedure might look like this:

| $\cancel{|}$ *(So far, we've kept 1 of 2 pieces)*

| $\cancel{|}$ *(So far, we've kept 2 of 4 pieces)*

| $\cancel{|}$ *(So far, we've kept 3 of 6 pieces)*

| $\cancel{|}$ *(Finally, we've kept 4 of 8 pieces)*

We've also given away 1 of 2 pieces.

And we've given away 4 of the 8 pieces.

If we look at the columns rather than at the rows we can see that we've divided 8 by 2. The fact that there are 2 columns tells us that we're dividing by 2. The fact that there are 4 in each column tells us that 2 X 4 = 8 or equivalently, 8 ÷ 2 = 4.

The fact that we had 8 piec here is not important. If there were 500 pieces, we'd still continue with "1 for me, 1 for you" until the 500 pieces were equally shared. We wouldn't even have to know that each youn ster got 250 pieces.

We have a special notation to indicate that we've divided an amount into 2 equal parts and taken 1 of these 2 parts.

We write the number of equal portions (in this case, 2). Directly above it we write the number of these portions we take (in this case, 1). And we separate the two numbers by a horizontal line (called a "bar"). So in this discussion, we'd write $\frac{1}{2}$ which we read as *"one-half". In our illustration we've shown that $\frac{1}{2}$ of 8 is 4. More generally, to find $\frac{1}{2}$ of any whole number, we divide that whole number by 2.*

Because of how it looks we often read $\frac{1}{2}$ as "1 __over__ 2"

__Example 1__

How much is $\frac{1}{2}$ of 84?

Answer: 42

By definition $\frac{1}{2}$ of 84 means that we divide 84 by 2. From Module 3 we know that $84 \div 2 = 42$, so the answer is 42.

In terms of dividing candy, if the youngsters equally shared 84 pieces of candy by the "1 for me, 1 for you...." process, each youngster would have gotten 42 pieces. Dividing 84 by 2 is a quicker (but not as intuitive) way to get the answer.

Example 1 would become more complicated if we wanted to find half of a number that was not a multiple of 2. We'll come to grips with this problem, but for now let's assume that we'll use only multiples of 2 in Example 1.

For example, half of 10 is 5 and half of 12 is 6. Hence half of 11 is more than 5 but less than 6. There are no whole numbers that are less than 6 but more than 5.

****

__Definition__

A whole number is called __even__ *if it's a multiple of 2.*

Otherwise, it's called __odd__.

So any whole number is either even or odd--one or the other, but not both.

****

When we divide a whole number by 2 the remainder has to be either 0 or 1. If it's 0 the number is even. If it's 1 the number is odd. Even numbers end in 0,2,4,6, or 8. Odd numbers end in 1,3,5,7, or 9.

Sometimes we might want to divide an amount into three equal parts rather than two. In this case we talk about $\frac{1}{3}$ of a number. We read $\frac{1}{3}$ as <u>*one-third*</u>. To take $\frac{1}{3}$ of a number we divide the number by 3.

or as "1 over 3"

<u>Example 2</u>

How much is $\frac{1}{3}$ of 12?

Answer: 4

$\frac{1}{3}$ of 12 tells us to divide 12 by 3.

Since $12 \div 3 = 4$, the answer is 4.

In terms of sharing candy, suppose you share 12 pieces of candy equally with two friends. You then keep 1 of every 3 pieces, as you count "1 for me, 1 for you (1st friend), 1 for you (2nd friend)." In terms of tally marks:

| ~~|~~ ~~|~~ *(You've kept 1 out of 3)*

| ~~|~~ ~~|~~ *(Now it's 2 out of 6)*

| ~~|~~ ~~|~~ *(Now it's 3 out of 9)*

| ~~|~~ ~~|~~ *(Now it's 4 out of 12)*

The 3 columns tell us we've divided by 3. The four in the column not crossed out tells us you've kept 4. That is, one-third of 12 is

If we look at our solution to Example 2, we notice that we could have tried to describe the portion we gave away rather than the portion we kept. In this case, we've still divided the total amount into three equal parts, *but this time we're talking about the <u>2</u> parts we've given away. That is, we've kept <u>1</u> third, but we've given away <u>2</u> thirds.*

In this discussion we'll assume that the number we'r dividing by 3 is a multiple of 3.

We now generalize our notation. Since we're still talking about dividing a number by 3, we write the 3 on the bottom. But because we're taking 2 of these 3 equal parts, we write a 2 above the 3, separated

by a bar. That is:

$$\frac{2}{3}$$

which we read as 2 *thirds*.

Notice that 2 is now modifying the noun "thirds"

To take $\frac{2}{3}$ of a whole number, we divide it by 3 and then multiply the quotient by 2.

Example 3

How much is $\frac{2}{3}$ of 600?

Answer: 400

The instructions tell us to divide 600 by 3 and to multiply this quotient by 2.

$600 \div 3 = 200$

$200 \times 2 = 400$

So the answer is 400.

Note that we'd get the same answer if we first multiplied 600 by 2 and then divided this product by 3

$600 \times 2 = 1{,}200$

$1{,}200 \div 3 = 400$

While order isn't important, it is important to remember that we divide by the 3 and multiply by the 2.

In terms of candy and tally marks, if we put the tally marks in groups (rows) of 3, we get 200 rows. In each row we give away 2 pieces of candy (tally marks). So all in all we give away 2 tally marks 200 times--or a total of 400 tally marks.

****

New Vocabulary

(1) The expression $\frac{2}{3}$ is called a <u>common fraction</u>

(2) The "bottom" number is called the <u>denominator</u>. *("Denominator" suggests "denomination or size)*

(3) The "top" number is called the <u>numerator</u>. *("Numerator" suggests "ennumerate" or "count")*

****

We're using $\frac{2}{3}$ only for the purpose of illustration.

So in 2/3, 3 is the denominator. We divide by the denominator to find the <u>*size*</u> *of each portion.*

In 2/3, 2 is the numerator. It tells us to take 2 (of the 3) equal portions.

<u>Example 4</u>

In the common fraction $\frac{3}{5}$ which number is:

(a) the numerator? (b) the denominator?

Answer: (a) 3 (b) 5

(a) The numerator is the "top" number.

More importantly, it tells us how many portions we're taking.

(b) The denominator is the "bottom" number.

More importantly, it is the number we divide by to find the size of each of the equal portions.

<u>Example 5</u>

How much is $\frac{3}{5}$ of 60?

(In words, how much is 3 fifths of 60)

Answer: 36

We divide 60 by 5 to get 12. We then multiply 12 by 3 to get 36.

That is, 60 can be divided into 5 equal portions, each with 12. If we then take 3 of these portions, we're taking 36 altogether:

X X X X X X X X X X X X
X X X X X X X X X X X X
X X X X X X X X X X X X
| | | | | | | | | | | |
| | | | | | | | | | | |

(We've taken 3 of the 5 rows or 3 out of 5 in each colum

In terms of *rates*, we can say that at the rate of 3 out of each 5, we'd take 36 out of each 60. More generally, we have the following:

3 out of each 5 means that at this same rate, we'd take:

3	out of each 5
6	out of each 10
9	out of each 15
12	out of each 20
15	out of each 25
18	out of each 30
	and so on

The first number is a multiple of 3. The second is the corresponding multiple of 5. For example, 60 is the 12th multiple of 5. The 12th multiple of 3 is 36. So at a rate of 3 out of 5, we take 36 out of 60.

In terms of common fractions, the above table shows us that $\frac{3}{5}$, $\frac{6}{10}$, $\frac{9}{15}$, $\frac{12}{20}$, $\frac{15}{25}$, $\frac{18}{30}$, . . . all name the same ratio. For example:

For type-setting purposes we often write 3/5, 6/10, 9/15 and so on.

Example 6

How much is $\frac{9}{15}$ of 60?

Answer: 36

This time the denominator is 15 and the numerator is 9. So we divide 60 by 15 to get 4 and then we multiply 4 by 9 to get 36.

It is not a coincidence that Examples 5 and 6 have the same answer.

$\frac{9}{15}$ and $\frac{3}{5}$ do not look alike. In fact they consist of different pairs of numbers. Yet they name the same ratio. That is, at a rate of 3 out of 5, you'd take 9 out of 15. For this reason we say that $\frac{9}{15}$ and $\frac{3}{5}$ are *equivalent* and we write $\frac{9}{15} = \frac{3}{5}$

2 X 12 and 4 X 6 do not look alike. They are different numerals that name the same number (24). It's analogous to saying that "Mark Twain" and "Samuel Clemens" are not the same name but that they name the same person.

Key Property

If we multiply the numerator and denominator of any common fraction by the same non-zero number we get an equivalent common fraction.

For example if we start with 3/5 and multiply numerator and denominator by 3 we get the equivalent common fraction 9/15

Example 7

Find a common fraction whose numerator is 24 that is equivalent to the common fraction: $\frac{4}{7}$

Answer: $\frac{24}{42}$

What we're really asking for is an answer to the ratio:

$$\frac{4}{7} = \frac{24}{?}$$

Since 24 is the 6th multiple of 4, we see that we have to multiply 4 by 6 to get 24. Then since we multiplied the numerator of $\frac{4}{7}$ by 6 we must also multiply the denominator of $\frac{4}{7}$ by 6 to obtain an equivalent common fraction. We get:

Another way of viewing this example is that 4/7 of 42 i 24. That is we divide 42 b 7 to get 6; and then we mul iply 6 by 4 to get 24.

$$\frac{4}{7} = \frac{4 \text{ X } 6}{7 \text{ X } 6} = \frac{24}{42}$$

What this means is that a rate of 4 out of every 7 is the same as 24 out of every 42. *We usually prefer to say "4 out of 7" because it is easier to visualize that "24 out of 42"*

If we <u>add</u> 6 to numerator an denominator we do change th ratio. For example 4 out o every 7 is not the same rat as 10 (that is, 4 + 6) out of every 13 (that is, 7 + 6 Indeed, 4 out of every 7 would be 8 out of each 14-- which is much less than 10 out of each 13.

Example 8

Find a common fraction whose denominator is 35 that is equivalent to the common fraction: $\frac{4}{7}$

Answer: $\frac{20}{35}$

In this case we have: $\frac{4}{7} = \frac{?}{35}$

Looking at the denominators we see that 35 is the 5th multiple of 7. So we multiply both numerator and denominator of $\frac{4}{7}$ by 5 to obtain:

$$\frac{4}{7} = \frac{4 \text{ X } 5}{7 \text{ X } 5} = \frac{20}{35}$$

4 out of 7 is the same rate as 20 out of 35. Namely:

ł ł ł ł | | | (4 out of 7)
ł ł ł ł | | | (8 out of 14)
ł ł ł ł | | | (12 out of 21
ł ł ł ł | | | (16 out of 28
ł ł ł ł | | | (20 out of 35

(35 is the 5th multiple of and 20 is the 5th multiple of 4)

Example 9

Find a common fraction whose denominator is 35 that is equivalent to the common fraction:
$$\frac{3}{5}$$

Answer: $\frac{21}{35}$

In this case we have $\frac{3}{5} = \frac{?}{35}$

Looking at the two denominators we see that 35 is the 7th multiple of 5. So we multiply numerator and denominator of $\frac{3}{5}$ by 7 to get:

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35}$$

In other words, at a rate of 3 out of each 5, we'd have 21 out of each 35.

Check: $\frac{3}{5}$ *of 35 is*
$(35 \div 5) \times 3$ or
7×3 or 21

Examples 8 and 9 give us a hint as to how to compare the size of two or more ratios.

Example 10

Based on Examples 8 and 9, which common fractions names the greater ratio--
$$\frac{3}{5} \text{ or } \frac{4}{7}$$

Answer: $\frac{3}{5}$

In Example 8 we saw that $\frac{4}{7}$ of 35 was 20 and in Example 9 we saw that $\frac{3}{5}$ of 35 was 21.

In the language of common fractions:

$\frac{4}{7} = \frac{20}{35}$ or 20 thirty-fifths

and $\frac{3}{5} = \frac{21}{35}$ or 21 thirty-fifths.

So $\frac{3}{5}$ exceeds $\frac{4}{7}$ by $\frac{1}{35}$

In looking at 3 fifths and 4 sevenths don't be led astray by the fact that 3 is less than 4. For example 3 feet is more than 4 inches and 3 dimes has more value than 4 nickels. In comparing size, we must look at the nouns as well as the adjectives.

It is sometimes easier to think in terms of money. At a rate of 3 for \$5 you'd get 21 for \$35, while at a rate of 4 for \$7 you'd get 20 for \$35. So 3 for \$5 is the better deal. How much better? Well for every \$35 you spent, you'd get one more object than had you bought 4 for \$7.

The important observation is that we could have done Example 10 even if we hadn't already done Examples

8 and 9 first. For one thing we could have found equivalent common fractions for $\frac{3}{5}$ and $\frac{4}{7}$ that had a common denominator. For example:

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{12}{20} = \frac{15}{25} = \frac{18}{30} = \frac{21}{35} = \frac{24}{40} = \ldots$$

Think whole numbers here! the numerators are the mult-iples of 3 and the denomina-tors are multiples of 5.

and

$$\frac{4}{7} = \frac{8}{14} = \frac{12}{21} = \frac{16}{28} = \frac{20}{35} = \frac{24}{42} = \ldots$$

Here the numerators are the multiples of 4 and the denom-inators are the multiples of 7.

Looking at the two lines of equalities, we see that 35 is the first denominator that appears in both. In other words:

$\frac{3}{5}$ = 21 thirty-fifths, $\frac{4}{7}$ = 20 thirty-fifths,

and 21 thirty-fifths exceeds 20 thirty-fifths by 1 thirty-fifth.

In this form, 20 and 21 are whole numbers *which modify the same noun (thirty-fifth*

A quicker way to obtain this same result is to notice that the product of any two numbers is a multiple of each factor. For example, in this problem the denominators are 5 and 7, so 5 X 7 or 35 is a common multiple of both 5 and 7. In fact, 35 is the 5th multiple of 7 and the 7th multiple of 5. Remembering that we can multiply numerator and denominator of any common fraction by the same non-zero whole number, we have:

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35}$$

and

$$\frac{4}{7} = \frac{4 \times 5}{7 \times 5} = \frac{20}{35}$$

When the denominators are quite large, this technique is less frustrating than when we list all the equi-valent fractions. The list can become quite long, as we shall indicate in the next example.

Example 11

Which common fraction names the greater ratio, and by how much--

$\frac{4}{9}$ or $\frac{7}{16}$

Answer: $\frac{4}{9}$ *by* $\frac{1}{144}$

The two denominators are 9 and 16.

Their product is 9 X 16 or 144. 144 is the 16th multiple of 7 and the 7th multiple of 16.

With this in mind we have:

$$\frac{4}{9} = \frac{4 \times 16}{9 \times 16} = \frac{64}{144}$$

and

$$\frac{7}{16} = \frac{7 \times 9}{16 \times 9} = \frac{63}{144}$$

That is:

$\frac{4}{9}$ = 64 hundred-forty-fourths

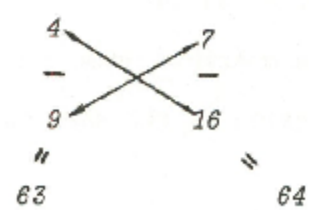
$\frac{7}{16}$ = 63 hundred-forty-forths

So $\frac{4}{9}$ exceeds $\frac{7}{16}$ by 1 hundred-forty-fourth (or $\frac{1}{144}$)

If we listed equivalent common fractions to find the common denominator we may have gotten discouraged before we finished. That is, 144 wouldn't occur until we listing the <u>16th</u> multiple of 9, or the <u>9th</u> multiple of 16. That is: 4/9 = 8/18= 12/27 = 16/36 = 20/45 = 24/54 etc and we're nowhere close to having 144 in the denominator.

Note

In the above procedure we essentially multiply the numerator of one common fraction by the denominator of the second. In this example, we had 4 X 16 or 64 and 7 X 9 or 63. This process is called <u>cross-multiplying</u>. *A quick way to find out whether two common fractions are equivalent is to cross-multiply. If the two products are equal, then the two common fractions are equivalent.*



In this example, the two cross-products were 63 and 64. This tells us that the two common fractions were not equivalent. But the fact that the difference between 63 and 64 is small tells us that the two ratios are rather similar.

We are now in a position to add or subtract common fractions based solely on our knowledge of whole numbers.

Example 12

How much is $\frac{3}{5} + \frac{1}{5}$?

Answer: $\frac{4}{5}$

We already know that if 3 and 1 modify the same noun, then 3 + 1 = 4. In this example, we have to add 3 *fifths* and 1 *fifth*. So the sum is 4 *fifths*, which we write as $\frac{4}{5}$.

In vertical form:

$$\begin{array}{rl} & 3 \textit{ fifths} \\ + & \underline{1 \textit{ fifth}} \\ & 4 \textit{ fifths} \end{array}$$

Again you may prefer to think in terms of money. $\frac{1}{5}$ *of a dollar is* $\frac{1}{5}$ *of 100 cents. Since* $100 \div 5 = 20$, $\frac{1}{5}$ *of a dollar is 20¢. So* $\frac{3}{5}$ *of a dollar is 20¢ X 3 or 60¢ and* $\frac{4}{5}$ *of a dollar is 20¢ X 4 or 80¢. So we have:*

$\frac{3}{5}$ *dollars +* $\frac{1}{5}$ *dollars =*

60 cents + 20 cents =

80 cents =

$\frac{4}{5}$ *dollars.*

You can also think in terms of time. An hour is 60 minutes. So 1/5 hours is 12 minutes, 3/5 hours is 12 X 3 or 36 minutes, and 4/5 hours is 12 X 4 or 48 minutes.

Hence: $\frac{3}{5}$ *hours +* $\frac{1}{5}$ *hours =*

36 mins + 12 mins =

48 mins =

$\frac{4}{5}$ *hours*

****

KEY POINT

To add two common fractions that have the same denominator, add the numerators and keep the common denominator. In symbols, if a, b, and c are whole numbers ($c \neq 0$)*:*

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

****

For example, don't say that $\frac{3}{5} + \frac{1}{5} = \frac{4}{10}$. *Just because a nickel and a nickel is a dime, we don't say that:*

$$\begin{array}{rl} & 3 \textit{ nickels} \\ + & \underline{1 \textit{ nickel}} \\ & 4 \textit{ dimes !} \end{array}$$

Remember we add the amounts and keep the common denomination.

If the denominators are different we rewrite the problem in terms of equivalent common fractions that have a common denominator. Once we've done this we proceed just as we did when the denominators were the same. For instance:

Example 13

How much is $\frac{3}{5} + \frac{1}{4}$?

We can only add like denominations.

Looking at 4 and 5 we see that 4 X 5 or 20 will be a common denominator. 20 is the 4th multiple of 5 and the 5th multiple of 4. Hence:

$$\frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20}$$

$$\frac{1}{4} = \frac{1 \times 5}{4 \times 5} = \frac{5}{20}$$

Replacing $\frac{3}{5}$ by $\frac{12}{20}$ and $\frac{1}{4}$ by $\frac{5}{20}$, we get:

$$\begin{array}{ccccc} \frac{3}{5} & + & \frac{1}{4} & = \\ \downarrow & & \Downarrow & \\ \frac{12}{20} & + & \frac{5}{20} & = \\ & \frac{17}{20} & & \end{array}$$

Here we can't add yet because we have different denominations.

Here the denominations are the same (common denominators) so we add the numerators and keep the common denominator.

Answer: $\frac{17}{20}$

Caution!!

$\frac{3}{5} + \frac{1}{4}$ *is* <u>*NOT*</u> $\frac{4}{9}$

For example 3/5 of a dollar is 60¢ and 1/4 of a dollar is 25¢. So 3/5 dollars + 1/4 dollars is 60¢ + 25¢ or 85¢, which is 17/20 of a dollar, not 4/9 of a dollar.

As a check notice the following. A fifth of an hour is 12 minutes so 3/5 hours is 36 minutes. A fourth of an hour is 15 minutes. A twentieth of an hour is 3 minutes (60 ÷ 20 = 3). Hence 17/20 of an hour is 3 X 17 or 51 minutes. Therefore:

$\frac{3}{5}$ *hours* + $\frac{1}{4}$ *hours* =

36 minutes + 15 minutes =

51 minutes =

$\frac{17}{20}$ *hours*

The fact that 36 minutes + 15 minutes = 51 minutes should also be true in the language of hours. Thus $\frac{3}{5}$ *(hours) +* $\frac{1}{4}$ *(hours)* <u>*MUST EQUAL*</u> $\frac{17}{20}$ *(hours)*

The Psychology of Using Common Fractions

If we divide a number by 5 and then multiply the product by 3, it seems that we are doing a typical problem from Module 3. But if we say "Find $\frac{3}{5}$ of a number" it seems that we are in a more advanced situation. Therefore it is helpful when you see $\frac{m}{n}$ to think of dividing by n and multiplying by m.

where m and n are whole numbers and n is not 0.

Historically, the fear of fractions was overcome by picking nouns to name fractional parts. For example, rather than say $\frac{7}{16}$ pounds we learned to say 7 ounces. We learn to say 5 inches rather than $\frac{5}{12}$ feet.

Somehow the word "ounce" doesn't suggest a fraction even though it means 1/16 of a pound.

We also learn to say 1 minute instead of $\frac{1}{60}$ hours. Revisiting the fact that $\frac{3}{5}$ of 60 is 36, the idea is this. $\frac{3}{5}$ of an hour strongly suggests fractions. But since 1 hour is 60 minutes, $\frac{1}{5}$ of an hour is $60 \div 5$ or 12 minutes. Hence $\frac{3}{5}$ of an hour is 12 minutes 3 times or 36 minutes. The word "minute" is a noun that names a denomination. Therefore the noun replaces the denominator of the common fraction. *The numerator now becomes the adjective that modifies the denomination. Hence we wind up with a whole number modifying a noun.*

We often say "3/5 of an hour" or "3/5 of 1 hour" rather than "3/5 hours"

In summary when we say 36 minutes, we are using a different language to describe $\frac{36}{60}$ or $\frac{3}{5}$ hours. In this vein, then, $\frac{3}{5}$ hours + $\frac{1}{4}$ hours = $\frac{17}{20}$ hours is simply a translation of 36 minutes + 15 minutes = 51 minutes into the "language" of common fraction.

So the psychology is essentially this: If you know that the sum of 36 minutes and 15 minutes is 51 minutes but you don't know that the sum of 3/5 hours and 1/4 hours is 17/20 hours, you have a language problem--NOT a mathematics problem!

The method that we used in Example 13 can become uite cumbersome with larger denominators.

Example 14

Find the sum of $\frac{5}{18}$ and $\frac{2}{27}$?

Answer: $\frac{171}{486}$ *or* $\frac{19}{54}$

If we use the method of Example 13, we see that a common multiple of 18 and 27 is 18 X 27 or 486. In fact, 486 is the 18th multiple of 27 and the 27th multiple of 18.

In any event, this leads to:

$$\frac{5}{18} = \frac{5 \times 27}{18 \times 27} = \frac{135}{486}$$

$$\frac{2}{27} = \frac{2 \times 18}{27 \times 18} = \frac{36}{486}$$

Once we have a common denominator we add the numerators (135 + 36 = 171) and keep the common denominator (486)

Hence:

$$\frac{5}{18} + \frac{2}{27} =$$

$$\frac{135}{486} + \frac{36}{486} = \frac{171}{486}$$

But suppose we had listed equivalent common fractions in the hope of finding a common denominator. We'd obtain:

$$\frac{5}{18} = \frac{10}{36} = \frac{15}{54} = \frac{20}{72} = \frac{25}{90} = \frac{30}{108} = \ldots$$

$$\frac{2}{27} = \frac{4}{54} = \frac{6}{81} = \frac{8}{108}$$

It seems conceptually easier to grasp taking 19 out of each 54 than to take 171 out of each 486.

Looking at the two lists we see that the first common multiple of the denominators we come to is 54. That is we could have replaced $\frac{5}{18}$ *by* $\frac{15}{54}$ *and* $\frac{2}{27}$ *by* $\frac{4}{54}$ *to conclude that*

$$\frac{5}{18} + \frac{2}{27} =$$

$$\frac{15}{54} + \frac{4}{54} = \frac{19}{54}$$

What Example 14 highlights is the fact that there are many common multiples of two or more numbers. While it is true that 486 is a common multiple of 18 and 27, it is also true that so is 54. More to the point, 54 is the *least* common multiple of 18 and 27. *Any other multiple of 54 will also be a common multiple of 18 and 27.*

0 is a multiple of all numbers; hence by "least" we mean the least non-zero multiple

It may not seem that obvious at first glance, but 486 is the 9th multiple of 54. Hence if we start with $\frac{19}{54}$ and multiply numerator and denominator by 9 we get:

From Module 3 we know how to show that $54 \times 9 = 486$

$$\frac{19}{54} = \frac{19 \times 9}{54 \times 9} = \frac{171}{486} \qquad (1)$$

Aside from the answers $\frac{19}{54}$ and $\frac{171}{486}$ we could have had:

$$\frac{19}{54} = \frac{19 \times 2}{54 \times 2} = \frac{38}{108}$$

$$\frac{19}{54} = \frac{19 \times 3}{54 \times 3} = \frac{57}{162}$$

and so on.

Note that if we read (1) from left to right, we notice that we multiplied both numerator and denominator by 9 to get $\frac{171}{486}$. But if we read (1) from right to left notice that we divided *numerator and denominator by 9 to get $\frac{19}{54}$.*

The process of replacing $\frac{171}{486}$ by the equivalent common fraction $\frac{19}{54}$ is known as reducing a common fraction to lowest terms.

So in analyzing Example 14, we discover two problems of interest that we should discuss.

(1) *Given two or more whole numbers, is there a convenient way to find their* least *common multiple?*

(2) *Given a common fraction, how can we recognize whether it is in lowest terms; and if it isn't how can we reduce it to lowest terms?*

Multiplying the numbers wil allways yield a common multiple, but not necessarily the least.

Reducing common fractions t lowest terms obviously simplifies the arithmetic w have to do in the problem.

The concept of *prime numbers* helps us to effectively answer both of these questions.

We begin our study of prime numbers by recalling that any whole number is a multiple of itself and 1. But some numbers can be factored only in this way.

That is, for any whole number n: n = n X 1

Example 15

Write 6 as the product of two whole numbers, neither of which is 1.

Answer: 6 = 3 X 2 or 2 X 3

From our single-digit multiplication table of Module 3, we know that 6 is the 3rd multiple of 2 and the 2nd multiple of 3. That is, 6 is either 2 X 3 or 3 X 2.

So while 6 = 1 X 6, it is also equal to 2 X 3.

Because multiplication has the commutative property we don't worry too much about whether we write 2 X 3 or 2 X 3. It is customary to write the lesser factor first.

Example 16

Can 2 be written as the product of two whole numbers, neither of which is 1?

Answer: No

Certainly, 2 is a multiple of both 1 and 2. But since any non-zero multiple of a number is at least as great as the number, 2 cannot be a multiple of any other whole number. That is, 2 cannot be a multiple of 3 because 3 is greater than 2.

In terms of the multiplication table, 2 appears as an entry only in the "1-table" and the "2-table".

Definition

(1) Any whole number, greater than 1, is called a prime number, *if its only factors are 1 and itself.*

(2) 1 is called a unit.

(3) Any other (non-zero) whole number is called a composite number.

Again remember that any whole number has 1 and itself as factors.

Let's review Examples 15 and 16 in terms of our new vocabulary.

Example 17

(a) Is 2 a prime number?

(b) Is 6 a prime number?

Answer: (a) Yes (b) No

(a) In Example 16 we saw that the only factors of 2 are 2 and 1. Hence 2 is by definition a prime number.

(b) In Example 15 we saw that in addition to 1 and itself, 6 had 2 and 3 as factors. Hence 6 is not a prime number. In fact, it is a composite number.

Notice that a number cannot be both prime and composite Indeed, except for 1, we defined a number to be composite if it wasn't a prime number.

Note

Since $6 = 2 \times 3$, we see that 6 is both a multiple of 2 and a multiple of 3. Both 2 and 3 are prime numbers. This result is true in general. Namely: Any composite number is a multiple of of two or more prime numbers. *This fact gives us a nice way to determine prime numbers.*

3 appears as an answer in the 1-table and 3-table, but not in the 2-table. Sc 3 is a prime number.

Let's show how we can find all prime numbers that are less than a given number. For example, suppose we want to know all the prime numbers that are no greater than 50.

Step 1: *List all the whole numbers from 1 through 50. Now begin the "weeding out" process by crossing out 1. Circle the next number (2). It must be a prime because 1 is the only smaller number on our list so far.* Now cross out every other multiple of 2.

Recall 1 is by definition not a prime. It is a unit.

We also know from Example 1 that 2 is a prime number.

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~

11 ~~12~~ 13 ~~14~~ 15 ~~16~~ 17 ~~18~~ 19 ~~20~~

21 ~~22~~ 23 ~~24~~ 25 ~~26~~ 27 ~~28~~ 29 ~~30~~

31 ~~32~~ 33 ~~34~~ 35 ~~36~~ 37 ~~38~~ 39 ~~40~~

41 ~~42~~ 43 ~~44~~ 45 ~~46~~ 47 ~~48~~ 49 ~~50~~

The numbers we've crossed out are multiples of 2. Hence they have 2 as a factor. Any number greater tl 2 that has 2 as a factor can't be a prime number because a prime number has only 1 and itself as factoı

Step 2: *Go back to the beginning of the list and circle the first number that has not yet been either crossed-out or circled. Circle this number (In this case, 3) and then cross out its multiples.*

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~
11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~
~~21~~ ~~22~~ 23 ~~24~~ 25 ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~
31 ~~32~~ ~~33~~ ~~34~~ 35 ~~36~~ 37 ~~38~~ ~~39~~ ~~40~~
41 ~~42~~ 43 ~~44~~ ~~45~~ ~~46~~ 47 ~~48~~ 49 ~~50~~

If the number were a multiple of a smaller prime number, it would already have been crossed-out. The fact that it hasn't been crossed-out means that it must be a prime number.

Ignore numbers that have already been crossed-out. For example, the fact that 6 = 2 X 3 means that 6 is eliminated both because it's a multiple of 2 and also a multiple of 3. But there's no need to cross it out twice.

Step 3: *Repeat the process, going back to the beginning of the list and circle the next number that has neither been previously circled nor crossed-out (5). Then cross-out all multiples of this number (that is, the multiples of 5).*

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~
11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~
~~21~~ ~~22~~ 23 ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~
31 ~~32~~ ~~33~~ ~~34~~ ~~35~~ ~~36~~ 37 ~~38~~ ~~39~~ ~~40~~
41 ~~42~~ 43 ~~44~~ ~~45~~ ~~46~~ 47 ~~48~~ 49 ~~50~~

Notice that this time we're crossing out the numbers that end in 0 or 5.

Step 4: *Repeat the process once again; this time circling 7 and then crossing-out all the multiples of 7.*

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~
11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~
~~21~~ ~~22~~ 23 ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~
31 ~~32~~ ~~33~~ ~~34~~ ~~35~~ ~~36~~ 37 ~~38~~ ~~39~~ ~~40~~
41 ~~42~~ 43 ~~44~~ ~~45~~ ~~46~~ 47 ~~48~~ ~~49~~ ~~50~~

That is, 7 is the first number that is neither circled nor crossed-out after Step 3.

Step 5: *We could continue the process begun in Steps 1 through 4, but there's no need to. Simply circle the remaining numbers. As explained in the margin, they are all prime numbers. For example if 47 were not a prime, one of its factors would have to be either 2, 3, 5, or 7. But every number that is a multiple of either 2, 3, 5, or 7 has already been crossed-out. In any event we have:*

7 X 7 = 49 which is less than 50 while 8 X 8 = 64 which is greater than 50. This means that when we factor any number on our list at least one of the factors must be less than 8. (If both were greater than 8, the product would be greater than 64) Any number less than 8 must be a multiple of either 2, 3, 5, or 7 (It could be a multiple of 4, but 4 is itself a multiple of 2) Since we've crossed-out all multiples of 2, 3, 5, and 7, the numbers that are left must themselves be primes.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~

So 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47 are the only prime numbers that are less than 50.

While our procedure may have seemed cumbersome, it was quite effective. It's major weakness is that to find whether a particular number is a prime number, we must investigate whether the numbers before it are also primes.

If we wanted the prime numbers that were less than 1,000 we'd use the same procedure except that our list would go from 1 through 1,000

For example to show that 47 was a prime number, we had to cross-out the multiples of 2, 3, 5, and 7.

Example 18

Is 2,500 a prime number?

Answer: No

2,500 is the 1,250th multiple of 2.

The fact that 2 is a factor of 2,500 means that 2,500 cannot be a prime number.

Key point

Look at the multiples of 2. Namely: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, ... They all end in 0, 2, 4, 6, or 8. Conversely, every number that ends in 2, 4, 6, 8, or 0 is a multiple of 2. Hence: Every whole number that ends in 0, 2, 4, 6, or 8 is a multiple of 2. Hence 2 is the only prime number that is even.

This means that except for 2, all the prime numbers are odd numbers. However, as we've already seen, not all odd numbers are prime numbers.

Example 19

Is 177 a prime number?

Answer: No

If 177 is not a prime number it must be divisible by a prime number. We already know that 2, 3, 5, 7, 11, ... are prime numbers. Let's see if any of these is a factor of 177.

In fact 13 X 13 = 169 and 14 X 14 = 196. Hence, one of the factors of 177 must be less than 14. The only prime numbers that are less than 14 are 2, 3, 5, 7, and 11. So these are the only ones we have to check.

Since 177 doesn't end in 0, 2, 4, 6, or 8 it doesn't have 2 as a factor. So let's try 3.

$$\begin{array}{r} 59 \\ 3\overline{)177} \\ -15 \\ \hline 27 \\ -27 \\ \hline 0 \end{array}$$

This shows us that 177 is the 59th multiple of 3. In particular, 177 is not a prime number because it has factors other than 1 and itself.

<u>A Little "Trick"</u>

To see whether a number is divisible by 3, we need only see whether the sum of its digits is divisible by 3. In this example 1 + 7 + 7 = 15 and 15 is the 5th multiple of 3. Since 15 is divisible by 3 so is 177.

This trick will be explained in the videotaped lecture.

In fact we could add the digits in 15 to get 1 + 5 = 6 and it is easy to see that 6 is divisible by 3.

<u>Example 20</u>

Is 159 a prime number?

Answer: No

Let's try out our "list" of prime numbers. First of all since 159 ends in 9 it can't be a multiple of 2. The sum of the digits gives us:

1 + 5 + 9 = 15 = 1 + 5 = 6, and

6 is divisible by 3. Hence 159 is divisible by 3. Since 159 has factors other than 1 and itself, it is not a prime number.

Check:

$$\begin{array}{r} 53 \\ 3\overline{)159} \\ -15 \\ \hline 9 \\ -9 \\ \hline 0 \end{array}$$

So 159 is the 53rd multiple of 3 or the 3rd multiple of 53.

Example 21

Is 139 a prime number?

Answer: Yes

11 X 11 = 121 and 12 X 12 = 144.

Since 139 is less than 144, one of its factors must be less than 12. Hence if 139 is not a prime number it must be divisible by at least one of the numbers: 2, 3, 5, 7, or 11.

(1) 139 ends in a 9 so it can't be divisible by 2.

Get the idea? We don't have to see whether 139 is divisible by 8 because as soon as it's not divisible by 2 it can't be divisible by 8 (because 8 is divisibl by 2).

(2) 1 + 3 + 9 = 13 = 1 + 3 = 4. Since 4 is not divisible by 3, neither is 139.

(3) 139 can't be divisible by 5, because all multiples of 5 end in either 0 or 5.

See why we only have to check prime numbers? Every composite number is a multiple of at least one prime number (If it weren't, it would be a prime number itself)

(4) Let's divide 139 by 7:

$$\begin{array}{r}
19 \\
7\overline{)139} \\
-7 \\
\hline
69 \\
-63 \\
\hline
6
\end{array}$$

Since the remainder is not 0, 139 is not divisible by 7.

(5) Finally we check to see whether 139 is divisible by 11.

$$\begin{array}{r}
12 \\
11\overline{)139} \\
-11 \\
\hline
29 \\
-22 \\
\hline
7
\end{array}$$

Since the remainder is not 0, 139 is not divisible by 11. Hence 139 is a prime number.

Prime numbers play a major role in factoring. For example, while 24 can be factored in several ways, there is only one way to write 24 as a product in which each factor is a prime number.

For example: 12 X 2, 8 X 3, 6 X 4, 24 X 1, 2 X 3 X 4

Example 22

Write 24 as a product, in which each factor is a prime number.

Answer: 2 X 2 X 2 X 3

Suppose we start with 24 = 2 X 12.

In this form, 2 is a prime number, but 12 isn't. But we can rewrite 12 as 2 X 6.

Hence:

$$\begin{aligned} 24 &= 2 \times 12 \\ &= 2 \times (2 \times 6) \\ &= 2 \times 2 \times 6 \end{aligned}$$

Multiplication is associative so we don't need the grouping symbols. We used them just to remind us that we replaced 12 by 2 X 6

While 2 is a prime number, 6 isn't. 6 factors into 2 X 3. Therefore:

$$\begin{aligned} 24 &= 2 \times 12 \\ &= 2 \times 2 \times 6 \\ &= 2 \times 2 \times (2 \times 3) \\ &= 2 \times 2 \times 2 \times 3 \end{aligned}$$

We can "stack" the factors.

$$\begin{array}{r|l} 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline & 3 \end{array}$$

That is, we divided 24 by 2 to get 12. Then we divided 12 by 2 to get 6. Then we divided 6 by 2 to get 3. The process ends when the quotient is a prime number. What the "stack" shows is that 24 = 2 X 2 X 2 X 3

Note:

No matter what factors we started with the answer would remain the same. For example:

$$\begin{aligned} 24 &= 8 \times 3 \\ &= (4 \times 2) \times 3 \\ &= 4 \times 2 \times 3 \\ &= (2 \times 2) \times 2 \times 3 \\ &= 2 \times 2 \times 2 \times 3 \end{aligned}$$

It is generally agreed, unless otherwise stated, to list the prime factors from the smallest to the greatest. Once this is done, there is only one way to write a whole number as a product of prime numbers.

So we wouldn't write $2 \times 3 \times 2 \times 2$

Note:

This is another reason why we don't want 1 to be a prime. Namely, if it were, there would be an endless number of ways to factor a number into prime factors. For example:

$$24 = 1 \times 2 \times 2 \times 2 \times 3$$

$$24 = 1 \times 1 \times 2 \times 2 \times 2 \times 3$$

Multiplying by 1 doesn't change the product.

Sometimes we use exponential notation for brevity. In the same way that 2 X 3 abbreviates the *sum* of 3 two's, we write 2^3 to abbreviate the *product* of 3 two's.

In this context 10^3 has the same meaning as in Module 1 Namely, 10^3 is now the product of 3 tens or: $10 \times 10 \times 10$ which is 1,000 This agrees with our earlier definition of a 1 followed by 3 zeros. That is, whenever we multiply by 10 we annex a zero.

So rather than write:

$$24 = 2 \times 2 \times 2 \times 3$$

we can write instead:

$$24 = 2^3 \times 3$$

Let's make sure you understand this new notation so that you can use it as shorthand.

Example 23

What number is named by 5^3 ?

Answer: 125

5^3 means the product of 3 fives or $5 \times 5 \times 5$. $5 \times 5 = 25$, and $25 \times 5 = 125$. Therefore 5^3 is 125.

Example 24

What number is named by 3^5?

Answer: 243

Be careful of the order. 3^5 denotes the product of 5 threes or $3 \times 3 \times 3 \times 3 \times 3$.

$3 \times 3 = 9$, $9 \times 3 = 27$, $27 \times 3 = 81$

and $81 \times 3 = 243$. Hence $3^5 = 243$.

$3 \times 3 \times 3 \times 3 \times 3$ → (9) (27) (81) (243)

General Notation

If b is any whole number and n is any non-zero whole number, then

$$b^n$$

denotes the product of n b's.

b is called the base.

n is called the exponent

and b^n is called the nth power of b, or b to the nth power.

So in 4^6, 4 is the base and 6 is the exponent. The number named is the product of 6 fours or: $4 \times 4 \times 4 \times 4 \times 4 \times 4$

But 6^4 stands for the product of 4 sixes or: $6 \times 6 \times 6 \times 6$

Be careful in using exponents. As we can see from Examples 23 and 24, order makes a difference. While 5×3 is equal to 3×5, 3^5 and 5^3 are quite different.

Example 25

Write 168 as a product of prime numbers.

Answer: $2 \times 2 \times 2 \times 3 \times 7$ or $2^3 \times 3 \times 7$

Because 168 ends in 2 we know that it is divisible by 2. This gives us:

$$\begin{array}{r} 2\underline{)1\ 6\ 8} \\ 8\ 4 \end{array} \quad (Hence\ 168 = 2 \times 84)$$

Ordinarily we'd write

$$\begin{array}{r} 8\ 4 \\ 2\overline{)1\ 6\ 8} \end{array}$$

but the other way makes it easier to "stack" the factors.

84 ends in 4 so it, too, is divisible by 2. This gives us:

$$\begin{array}{r} 2\underline{)1\ 6\ 8} \\ 2\underline{)8\ 4} \\ 4\ 2 \end{array} \quad (Hence\ 168 = 2 \times 2 \times 42)$$

We'll keep going until every factor is a prime number.

42 ends in 2, so it, too, is divisible by 2.

Remember, we keep working with the smallest prime numbers and "work up"

Hence:

2)1 6 8
 2)8 4
 2)4 2
 2 1 *(Hence 168 = 2 X 2 X 2 X 21)*

21 ends in a 1 so it is not divisible by 2. However the sum of the digits, 2 + 1, is 3 and this is divisible by 3. Hence 21 is divisible by 3. In fact 21 = 3 X 7. Hence:

2)1 6 8
 2)8 4
 2)4 2
 3)2 1
 7 *(Hence 168 = 2 X 2 X 2 X 3 X 7)*

Each of the divisors is a prime divisor of 168.

Since 7 is a prime number, the process stops. Thus the prime factorization of 168 is 2 X 2 X 2 X 3 X 7 or 2^3 X 3 X 7.

The process of writing a number as the product of prime numbers is called PRIME FACTORIZATION.

Example 26

Write 180 as a product of prime numbers.

Answer: 2 X 2 X 3 X 3 X 5
or
2^2 X 3^2 X 5

Since 180 ends in a zero it is divisible by 2. We get:

2)1 8 0
 9 0 *(Hence 180 = 2 X 90)*

90 ends in 0, so 90 is also divisible by 2:

2)1 8 0
 2)9 0
 4 5 *(Hence 180 = 2 X 2 X 45)*

45 is not divisible by 2, but it is divisible by 3 since 4 + 5 = 9 and 9 is divisible by 3. Hence:

2)1 8 0
 2)9 0
 3)4 5
 1 5 *(Hence 180 = 2 X 2 X 3 X 15)*

We can begin with any prime factor we wish. For example 180 ends in 0 so it is divisible by 5.

5)1 8 0
3 6

3 + 6 = 9, so 36 is divisib by 3:

5)1 8 0
3)3 6
2)1 2
2)6
3

So 180 = 5 X 3 X 2 X 2 X 3 and if we reorder the factors from smallest to greatest we have:
2 X 2 X 3 X 3 X 5

Since $1 + 5 = 6$ and 6 is divisible by 3, 15 is divisible by 3. This gives us:

```
2)1 8 0
  2)9 0
  3)4 5
  3)1 5
      5
```

(Hence $180 = 2 \times 2 \times 3 \times 3 \times 5$)

Since 5 is a prime number, the prime factorization is complete.

Now we come to the main point. The results of Examples 25 and 26 allows us to reduce a common fraction to lowest terms. *The thing to remember is that if the same factor occurs in both the numerator and the denominator of a common fraction we can "cancel" it without changing the ratio named by the fraction.*

For example:

$$\frac{14}{16} = \frac{7 \times 2}{8 \times 2} = \frac{7}{8}$$

In dividing numerator and denominator by 2, we, in effect, "crossed-out" or "cancelled" the 2's.

Let's see how this works.

Example 27

Reduce $\frac{168}{180}$ to lowest terms.

Answer: $\frac{14}{15}$

That is, to take 168 out of each 180 is the same rate as taking 14 out of each 15.

By prime factorization (as done in Examples 25 and 26) we have:

$$\frac{168}{180} = \frac{2 \times 2 \times 2 \times 3 \times 7}{2 \times 2 \times 3 \times 3 \times 5}$$

Get the idea? If we hadn't already written 168 and 180 as products of prime numbers, we now would have.

We can cancel two factors of 2 in the numerator with two factors of 2 in the denominator and we can cancel a factor of 3 from both numerator and denominator to get:

$$\frac{168}{180} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3} \times 7}{\cancel{2} \times \cancel{2} \times 3 \times \cancel{3} \times 5}$$

We can't cancel the 3rd 2 in the numerator because there are no more 2's in the denominator. We can only cancel common factors.

Since no other common factors remain the problem is finished. $2 \times 7 = 14$ and 3×5

is 15. So the final answer is:

$$\frac{168}{180} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3} \times 7}{\cancel{2} \times \cancel{2} \times 3 \times \cancel{3} \times 5} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}$$

Check:

$$\frac{14}{15} = \frac{14 \times 12}{15 \times 12} = \frac{168}{180}$$

(We knew that we had to multiply by 12 because the factors we cancelled were 2, 2, and 3--and 2 X 2 X 3 = 12

Note

This technique requires that the factors be prime. For example if we had:

$$\frac{4 \times 9}{3 \times 6}$$

no factor in the numerator looks the same as any factor in the denominator. However when we replace 4 by 2 X 2, 9 by 3 X 3 and 6 by 2 X 3, we get:

$$\frac{(2 \times 2) \times (3 \times 3)}{3 \times (2 \times 3)}$$

Reordering the factors give us:

$$\frac{2 \times \cancel{2} \times \cancel{3} \times \cancel{3}}{\cancel{2} \times \cancel{3} \times \cancel{3}}$$

The extra bonus is that prime factorization also helps us find the least common multiple of two or more numbers. The idea is something like this. Suppose we want the least common multiple of 4 and 6. We know that the prime factorization of 4 is 2 X 2 and that the prime factorization of 6 is 2 X 3. What does this mean?

Well, it means that any multiple of 4 must contain 2 X 2 as part of its prime factorization and that any multiple of 6 must contain 2 X 3 as part of its prime factorization. So suppose we look at 4 first. We know that any multiple of 4 contains 2 x 2.

That is, any multiple of 4 has the form: 4 X ___ or 2 X 2 X ___

That is, 6 X __ = 2 X 3 X _

Therefore we start with 2 X 2 as our "trial" least common multiple of 4 and 6. Next we look at 6 and see that any multiple of it must contain 2 X 3. But the 2 is already contained in 2 X 2. So all we have to tack on to 2 X 2 is a factor of 3. This leads to 2 X 2 X 3 or 12.

2 X 2 X 3 is divisible by 2 X 2. Infact, if we cancel the two 2's we see the quotient is 3. Similarly 2 X 2 X 3 is also divisible by 2 X 3. The quotien is 2.

No number less than 12 can be a common multiple of 4 and 6 because if we omitted a 2 in 2 X 2 X 3 the number would not be divisible by 4, and if we omitted the 3 the number would not be divisible by 3.

Look at the even numbers that aren't divisible by 4. They are 2, 6, 10, 14, 18,.. These numbers have but one 2 in their prime factorization.

Example 28

Find the least common multiple of 24 and 90.

Answer: 360

By prime factorization we have:

$$\begin{array}{r} 2\underline{)24} \\ 2\underline{)12} \\ 2\underline{)6} \\ 3 \end{array}$$

Therefore 24 = 2 X 2 X 2 X 3.

$$\begin{array}{r} 2\underline{)90} \\ 3\underline{)45} \\ 3\underline{)15} \\ 5 \end{array}$$

Hence 90 = 2 X 3 X 3 X 5.

So we can start with 24 = 2 X 2 X 2 X 3 to conclude that any multiple of 24 must contain 2 X 2 X 2 X 3 in its prime factorization.

Any multiple of 90 must contain 2 X 3 X 3 X 5 in its prime factorization. The 2 and one of the 3's is included in 2 X 2 X 2 X 3. So we need only annex the factors 3 and 5. This gives us:

2 X 2 X 2 X 3 X 3 X 5
(4) (8) (24) (72) (360)

2 X 2 X 2 X 3 "takes care of" 24 while 2 X 3 X 3 X 5 "takes care of" 90.

So 360 is a common multiple of 24 and 90. If we ommited one of the 2's the resulting product would not be divisible by 24. If we omitted a 3 or 5, the resulting product wouldn't be divisible by 90. So 360 is the least common multiple of 24 and 90.

360 = 24 X 15;so 360 is the 15th multiple of 24.

360 = 90 X 4; so 360 is the 4th multiple of 90.

Example 29

Find the sum of $\frac{5}{24}$ and $\frac{7}{90}$, making sure that your answer has been reduced to lowest terms.

Answer: $\frac{103}{360}$

We need a common multiple of 24 and 90.

Method 1

24 X 90 = 2,160. 2,160 is the 24th multiple of 90 and the 90th multiple of 24.

m X n is always a common multiple of m and n. It is the nth multiple of m and the mth multiple of n. However, m X n may not be the least common multiple of m and n.

So as we've done before in this module:

$$\frac{5}{24} = \frac{5 \times 90}{24 \times 90} = \frac{450}{2,160}$$

$$\frac{7}{90} = \frac{7 \times 24}{90 \times 24} = \frac{168}{2,160}$$

Hence:

$$\frac{5}{24} + \frac{7}{90} =$$

$$\frac{450}{2,160} + \frac{168}{2,160} =$$

$$\frac{618}{2,160} \qquad (1)$$

Remember that once we have have common denominators, we add by adding the numerators and keeping the common denominator.

Since 618 ends in 8 and 2,160 ends in 0, both 618 and 2,160 are divisible by 2. We can cancel the 2 from numerator and denominator of (1) to get:

618 = 2 X 309

2,160 = 2 X 1,080

$$\frac{309}{1,080} \qquad (2)$$

1,080 is divisible by 2 but 309 isn't. Hence 2 is not a common factor; and so it can't be cancelled.

3 + 0 + 9 = 12, which is divisible by 3.

1 + 0 + 8 + 0 = 9, which is divisible by 3.

So we can cancel 3 from both the numerator and denominator of (2) to get:

309 = 3 X 103

1,080 = 3 X 360

$$\frac{103}{360} \qquad (3)$$

Since 103 is a prime number, there is no need to factor 360. Namely, we can only cancel a common factor and 103 has no factors other than 1 and 103; and it is clear that 103 can't be a factor of 360.

11 X 11 = 121, so at least one factor of 103 must be less than 10. The only prime numbers less than 10 are 2, 3, 5, and 7 and none of these is a factor of 103.

The multiples of 103 are: 103, 206, 309, 412, ... and this shows us that 360 is not a multiple of 103.

The reason we had to do so much reducing was because we didn't start with the least common multiple of 24 and 90. This brings us to:

Method 2

Find the least common multiple of 24 and 90, which we did in the previous example. By way of review:

If we hadn't already found the least common multiple, we would do it now.

$$360 = 24 \times 15$$
$$360 = 90 \times 4$$

Hence:

$$\frac{5}{24} = \frac{5 \times 15}{24 \times 15} = \frac{75}{360}$$

$$\frac{7}{90} = \frac{7 \times 4}{24 \times 15} = \frac{28}{360}$$

$$\frac{5}{24} + \frac{7}{90} = \frac{75}{360} + \frac{28}{360} = \frac{103}{360}$$

If you tried to find a common multiple of 24 and 90 by listing the multiples of each, you'd have to have enough patience to get to the 15th multiple of 24.

As a practical application of this example, suppose you spent $\frac{5}{24}$ of your salary on food and an additional $\frac{7}{90}$ of your salary on medicines. Then this would account for \$103 out of every \$360 you earned. From another point of view, this would leave you with \$257 out of every \$360 you earned.

That is, \$360 - \$103 = \$257

We subtract in precisely the same way we added.

Namely:

Example 30

Write $\frac{5}{24} - \frac{7}{90}$ as a common fraction in lowest terms.

Answer: $\frac{47}{360}$

From Example 29 we already know that

$$\frac{5}{24} = \frac{75}{360} \text{ and } \frac{7}{90} = \frac{28}{360}$$

Hence:

$$\frac{5}{24} - \frac{7}{90} = \frac{75}{360} - \frac{28}{360} = \frac{47}{360}$$

When we subtract common denominations, we subtract the numerators ($75 - 28 = 47$) and keep the common denomination (360)

This is enough material for our first lesson on common fractions. In the next module we shall give still other motivations for inventing common fractions--and we shall discuss the operations of multiplication and division of rational numbers using common fractions. In the meantime, use the Study Guide to ensure that you have learned the material in this module.